Hilbert Norms For Graded Algebras

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Abstract

This paper presents a solution to a problem from superanalysis about the existence of Hilbert-Banach superalgebras. Two main results are derived:
1) There exist Hilbert norms on some graded algebras (infinite-dimensional superalgebras included) with respect to which the multiplication is continuous.
2) Such norms cannot be chosen to be submultiplicative and equal to one on the unit of the algebra.

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1 Introduction

The type of norms investigated in this article are generalizations of norms used for the symmetric tensor algebra in the white noise analysis [7][11] or in the Malliavin calculus [20]. But now more general algebras are included, especially the algebra of antisymmetric tensors (Grassmann algebra) and $\mathbb{Z}_2$-graded algebras (superalgebras) related to supersymmetry and to quantum probability [15].

A locally convex commutative superalgebra is a $\mathbb{Z}_2$-graded locally convex space $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ equipped with an associative continuous multiplication having the following property: for any $a, b \in \mathcal{E}_0 \cup \mathcal{E}_1$, $ab \neq 0$ the product satisfies $ab = (-1)^{p(a)p(b)} ba$ with the parity function p, which is defined on $(\mathcal{E}_0 \cup \mathcal{E}_1) \setminus \{0\}$ with $p(\mathcal{E}_0 \setminus \{0\}) = 0$, $p(\mathcal{E}_1 \setminus \{0\}) = 1$, and $p(ab) = |p(a) - p(b)|$. Typical examples are Grassmann algebras with finite or countable sets of generators. In superanalysis one considers modules over (commutative) superalgebras [16][8][5][19][17][4][18][10].[3] It is quite easy to define an infinite-dimensional Grassmann algebra with a non-Hilbertian norm [16]. But for a long time it was unknown whether the topology of a locally convex superalgebra - including the Grassmann algebra - can be defined with a Hilbert norm, and moreover, whether this norm can be chosen to be simultaneously submultilplicative and equal to one at the unit of the algebra. The paper gives a complete solution to these problems. Our theorems imply a positive answer to the first question and a negative answer to the second question.

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[3]In the pioneering works of Martin [14] and of Berezin [3] the Grassmann algebra itself has been used instead of these modules.

2 General considerations

Let $\mathcal{A}$ be an algebra over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with unit e_0. The product is denoted by $a, b \in \mathcal{A} \to ab \in \mathcal{A}$. We assume that $\mathcal{A}$ is provided with a positive definite inner product $a, b \in \mathcal{A} \to (a \mid b) \in \mathbb{K}$. The corresponding Hilbert norm $\|a\| = \sqrt{(a \mid a)} \geq 0$ is normalized at the unit $\|e_0\| = 1$. We are interested in such norms which allow a uniform estimate for the product of the algebra

$$\|ab\| \leq \gamma \|a\| \|b\| \tag{1}$$

with a constant $\gamma \geq 1$. In this section we prove under rather general conditions that this constant has the lower limit $\gamma \geq \sqrt{\frac{4}{3}}$.

Theorem 1 *Let $\mathcal{A}$ be an algebra over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with dimension* $\dim \mathcal{A} \geq 2$. *If this algebra satisfies the properties*
i)$\mathcal{A}$ is provided with a Hilbert inner product $(. \mid .)$ normalized at the unit e_0, $\|e_0\|^2 = (e_0 \mid e_0) = 1$,
ii)there exists at least one element $f \in \mathcal{A}, f \neq 0$, such that e_0, f and $f^2 = ff$ satisfy $(e_0 \mid f) = (f \mid f^2) = 0$ and $(e_0 \mid f^2) \geq 0$,
then the norm estimate $\|ab\| \leq \gamma \|a\| \|b\|$ is not valid for some $a, b \in \mathcal{A}$, if $\gamma < \sqrt{\frac{4}{3}}$.

Proof Since $f \neq 0$ we can normalize this element and assume $\|f\| = 1$. Take $a = e_0 + \lambda f$ with $\lambda \in \mathbb{R}$. Then $a^2 = e_0 + 2\lambda f + \lambda^2 f^2$ and
$\|a^2\|^2 = 1 + 2\lambda^2 (e_0 \mid f^2) + 4\lambda^2 + \lambda^4 \|f^2\|^2 \geq 1 + 4\lambda^2$. On the other hand $\|a\|^2 = 1 + \lambda^2$, and $\|a^2\|^2 \leq \gamma^2 \|a\|^4$ implies $1 + 4\lambda^2 \leq \gamma^2 (1 + \lambda^2)^2$. But this inequality is true for all $\lambda \geq 0$ only if $\gamma^2 \geq \sup_{\lambda \geq 0} (1 + 4\lambda^2)(1 + \lambda^2)^{-2} = \frac{4}{3}$. $\square$

This Theorem obviously applies to the tensor algebra $\mathcal{T} = \oplus_{n=0}^{\infty} \mathcal{T}_n$, where $\mathcal{T}_n$ is the subspace of tensors of degree n, and the norm is defined in the standard way as

$$\|f\|^2 = \sum_{n=0}^{\infty} w_n \|f_n\|_n^2 \text{ if } f = \sum_{n=0}^{\infty} f_n, \;\; f_n \in \mathcal{T}_n \tag{2}$$

with arbitrary positive weights $w_n > 0, n \in \mathbb{N}$ and $w_0 = 1$. In that case we can simply choose an element $f \in \mathcal{T}_1, f \neq 0$, to satisfy the assumptions with $(e_0 \mid f \otimes f) = 0$.

Theorem 1 can also be applied to a large class of algebras $\mathcal{A}$ which can be derived from the tensor algebra $\mathcal{T}$ by the following modifications of the product.

1. The product is generated by $f, g \in \mathcal{A}_1 = \mathcal{T}_1 \to f \circ g := f \otimes g + (-1)^\chi g \otimes f$ where $\chi = 0, 1 \bmod 2$ is a parity factor.

2. The product is generated by $f, g \in \mathcal{A}_1 = \mathcal{T}_1 \to$ $f \circ g := f \otimes g + (-1)^\chi g \otimes f + \omega(f, g) e_0$. Here χ is again a parity factor and $\omega(., .) : \mathcal{A}_1 \times \mathcal{A}_1 \to \mathbb{K}$ is a bilinear pairing.

The first class of algebras includes the algebra of symmetric tensors, the algebra of antisymmetric tensors (Grassmann algebra), and tensor products of these algebras, including

the $\mathbb{Z}_2$-graded algebras (superalgebras) used in quantum field theory. The assumptions of the Theorem 1 are satisfied for any non-vanishing element $f \in \mathcal{A}_1 = \mathcal{T}_1$.

The second class includes the Clifford product, the (symmetric) Wiener product, the antisymmetric Wiener product (with antisymmetric ω) and Le Jan's supersymmetric Wiener-Grassmann product [9][13][15]. In these cases the assumptions of Theorem 1 are satisfied if there exists a non-vanishing $f \in \mathcal{A}_1$ with $\omega(f,f) \geq 0$. Such a vector can always be found

if the algebra is complex, or

if the algebra is real and ω is not negative definite.

The last constraint is satisfied for the symmetric Wiener product on real spaces, and for the real Clifford system in quantum field theory [2]. In both cases the form ω is positive definite.

Moreover Theorem 1 is obviously true for any unital algebra $\mathcal{A}$, which has a nilpotent element f that is orthogonal to the unit element. If we only know that $\mathcal{A}$ has at least one nilpotent element, we can derive the weaker

Corollary 1 *Let $\mathcal{A}$ be an algebra which satisfies condition i) of Theorem 1. If this algebra has a nilpotent element f, then the norm estimate $\|ab\| \leq \|a\| \|b\|$ is not valid for some $a, b \in \mathcal{A}$.*

Proof We assume again $\|f\| = 1$. Then $a = e_0 + \lambda f$ with $\lambda \in \mathbb{R}$ and $a^2 = (e_0 + \lambda f)^2 = e_0 + 2\lambda f$ have the norms $\|a\|^2 = 1 + 2\lambda \mathrm{Re}\,(e_0, f) + \lambda^2$ and $\|a^2\|^2 = 1 + 4\lambda \mathrm{Re}\,(e_0, f) + 4\lambda^2$. If $\mathrm{Re}\,(e_0, f) = 0$ we can apply the arguments given in the proof for Theorem 1. If $\mathrm{Re}\,(e_0, f) = \gamma \neq 0$, then we chose $\lambda = -2\gamma$, and $\|a^2\|^2 = 1 + 8\gamma^2 \leq 1 = \|a\|^4$ is a contradiction. $\square$

3 Norm estimates for $\mathbb{Z}$-graded algebras

In this section we present Hilbert norm estimates for rather general $\mathbb{Z}$-graded algebras $\mathcal{A}$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.. We assume the following structure of $\mathcal{A}$.

1. The algebra is the direct sum $\mathcal{A} = \oplus_{n=0}^{\infty} \mathcal{A}_n$ of orthogonal spaces $\mathcal{A}_n$. Thereby $\mathcal{A}_0$ is the one dimensional space $\mathbb{K}$ spanned by the unit e_0 of the algebra. The product $a \circ b$ maps $\mathcal{A}_p \times \mathcal{A}_q$ into $\mathcal{A}_{p+q}$ for all $p, q \in \{0, 1, ...\}$.

2. The spaces $\mathcal{A}_n$ are provided with Hilbert norms $\|.\|_n$, $n = 0, 1, ...$The unit has norm $\|e_0\|_0 = 1$. The product of two homogeneous elements $a_p \in \mathcal{A}_p$ and $b_q \in \mathcal{A}_q$ satisfies
$$\|a_p \circ b_q\|_{p+q} \leq \|a_p\|_p \|b_q\|_q \tag{3}$$
if $a_p \in \mathcal{A}_p$ and $b_q \in \mathcal{A}_q$.

3. The algebra is provided with a family of Hilbert norms
$$\|a\|_{(\sigma)}^2 = \sum_{n=0}^{\infty} w_n(\sigma) \|a_n\|_n^2 \text{ if } a = \sum_{n=0}^{\infty} a_n,\ a_n \in \mathcal{A}_n \tag{4}$$

with $\sigma \in \mathbb{R}$. The factors $w_n(\sigma), n = 0, 1, ...$, are positive weights with the normalization $w_0(\sigma) = 1$ for all $\sigma \in \mathbb{R}$. The weights satisfy the inequalities $w_n(\sigma) \leq w_n(\tau)$ for all $n \in \mathbb{N}$ if $\sigma \leq \tau$.

An immediate consequence of these assumptions is $\|a\|_{(\sigma)} \leq \|a\|_{(\tau)}$ for all $a \in \mathcal{A}$ if $\sigma \leq \tau$. A simple example of such an algebra $\mathcal{A}$ is the tensor algebra $\mathcal{T}$. Its standard norm satisfies (3) with weights $w_n = 1$ for all $n = 0, 1....$ More interesting examples are the algebras of symmetric tensors or of antisymmetric tensors. With the notation $f \circ g$ for both the symmetric and the antisymmetric tensor product the estimate (3) is satisfied by the norms

$$\|f_1 \circ f_2 \circ ... \circ f_n\|_n^2 = \begin{cases} (n!)^{-1}\mathrm{per}(f_\mu \mid f_\nu) \text{ for symmetric tensors,} \\ (n!)^{-1}\det(f_\mu \mid f_\nu) \text{ for antisymmetric tensors,} \end{cases} \tag{5}$$

but it is violated if the factor $(n!)^{-1}$ is omitted. The standard norm[4] is defined without the factor $(n!)^{-1}$. In the notations used here it corresponds therefore to a norm (4) with a weight function $w_n = n!$.

Theorem 2 *If there exists a constant $\delta(\sigma, \tau, \rho) > 0$ such that the weight functions satisfy the inequalities*

$$(p + q - 1)w_{p+q}(\rho) \leq \delta(\sigma, \tau; \rho)w_p(\sigma)w_q(\tau) \text{ if } p, q \geq 1 \tag{6}$$

for values of σ, τ and ρ with $\sigma \leq \rho$ and $\tau \leq \rho$, then the product of $\mathcal{A}$ is estimated by

$$\|a \circ b\|_{(\rho)} \leq \gamma \cdot \|a\|_{(\sigma)} \|b\|_{(\tau)} \tag{7}$$

where the constant γ is $\gamma = \sqrt{3}\max(1, \delta(\sigma, \tau, \rho))$.

Proof For $a = a_0 + a_+$ and $b = b_0 + b_+$ with $a_0, b_0 \in \mathcal{A}_0 = \mathbb{K}$ and $a_+ = \sum_{n=1}^\infty a_n$, $b_+ = \sum_{n=1}^\infty b_n$ with $a_n, b_n \in \mathcal{A}_n, n \in \mathbb{N}$ the norm of $a \circ b$ is calculated by

$$\begin{aligned} &\|a \circ b\|_{(\rho)}^2 = \|a_0 b_0 + a_0 b_+ + a_+ b_0 + a_+ \circ b_+\|_{(\rho)}^2 \\ &\leq |a_0 b_0|^2 + 3\left(|a_0|^2 \|b_+\|_{(\rho)}^2 + \|a_+\|_{(\rho)}^2 |b_0|^2 + \|a_+ \circ b_+\|_{(\rho)}^2\right) \\ &\leq |a_0 b_0|^2 + 3\left(|a_0|^2 \|b_+\|_{(\rho)}^2 + \|a_+\|_{(\rho)}^2 |b_0|^2 + \sum_{n\geq 1} w_n(\rho) \left\|\sum\nolimits'_{p+q=n} a_p \circ b_q\right\|_n^2\right) \end{aligned}$$

The symbol $\sum'$ means summation with the constraint $p \geq 1, q \geq 1$. The sum $\sum_{p+q=n, p\geq 1, q\geq 1} ... = \sum'_{p+q=n} ...$ has $n - 1$ terms, hence
$\left\|\sum'_{p+q=n} a_p \circ b_q\right\|_n^2 \leq (n-1)\sum'_{p+q=n} \|a_p \circ b_q\|_n^2 \overset{(3)}{\leq} (n-1)\sum'_{p+q=n} \|a_p\|_p^2 \|b_q\|_q^2$.
If $w_n(\rho)$ is chosen such that (6) is satisfied we obtain
$\sum_{n\geq 1} w_n(\rho) \left\|\sum'_{p+q=n} a_p \circ b_q\right\|_n^2 \leq \delta \cdot \left(\sum_{p\geq 1} w_p(\sigma) \|a_p\|_p^2\right) \cdot \left(\sum_{q\geq 1} w_q(\tau) \|b_q\|_q^2\right)$

[4] The "standard" inner product of the symmetric/antisymmetric tensor algebra is characterized by the following property. Let $\mathcal{F}_i, i = 1, 2$, be two orthogonal subspaces of the space $\mathcal{A}_1$. Denote by $\mathcal{A}(\mathcal{F}_i)$ the subalgebra generated by elements $f \in \mathcal{F}_i$. Then $(a_1 \circ a_2 \mid b_1 \circ b_2) = (a_1 \mid b_1)(a_2 \mid b_2)$ holds for all $a_i \in \mathcal{A}(\mathcal{F}_i), i = 1, 2$.

$\leq \delta \|a_+\|^2_{(\sigma)} \|b_+\|^2_{(\tau)}$. For $\rho \leq \sigma, \tau$ we have in addition the inequalities $\|a_+\|^2_{(\rho)} \leq \|a_+\|^2_{(\sigma)}$ and $\|b_+\|^2_{(\rho)} \leq \|b_+\|^2_{(\tau)}$ such that finally

$$\begin{aligned} \|a \circ b\|^2_{(\rho)} &\leq |a_0 b_0|^2 + 3\left(|a_0|^2 \|b_+\|^2_{(\tau)} + \|a_+\|^2_{(\sigma)} |b_0|^2 + \delta \|a_+\|^2_{(\sigma)} \|b_+\|^2_{(\tau)}\right) \\ &\leq 3\gamma \|a\|^2_{(\sigma)} \|b\|^2_{(\tau)} . \end{aligned}$$

where γ is $\gamma = \max(1, \delta)$. □

As the first application of Theorem 2 we derive norms with respect to which the product of the algebra is continuous. In that case the inequality (6) has to be satisfied for identical weights $w_p(\sigma) = w_p(\tau) = w_p(\rho) = w_p,\ p \geq 1$. If we fix $q = 1$ then (6) implies $p \cdot w_{p+1} \leq \delta \cdot w_p \cdot w_1$ for $p \in \mathbb{N}$. As a consequence we obtain $w_p \leq \delta^{p-1} \left((p-1)!\right)^{-1} w_1, p \geq 1$. The slowest decrease of the weights which might be possible according to our estimates is therefore $w_p \sim \left((p-1)!\right)^{-1}$. The proof that such a solution actually exists follows from the simple estimate $\begin{pmatrix} m+n \\ m \end{pmatrix} = \frac{(m+n)!}{m!n!} \geq 1$ if $m, n \geq 0$. Hence $(p+q-1)\frac{1}{(p+q-1)!} = \frac{1}{(p+q-2)!} \leq \frac{1}{(p-1)!}\frac{1}{(q-1)!}$ is valid for all $p, q \geq 1$. Since

$$2^{m+n} \geq \begin{pmatrix} m+n \\ m \end{pmatrix} = \frac{(m+n)!}{m!n!} \geq m+n \text{ if } m, n \geq 1, \tag{8}$$

also $(p+q-1)\frac{1}{(p+q)!} < \frac{1}{(p+q-1)!} \leq \frac{1}{p!}\frac{1}{q!}$ follows for all $p, q \geq 1$. We have therefore derived

Corollary 2 *If the norm is defined with the weights $w_0 = 1,\ w_n = \left((n-1)!\right)^{-1},\ n \geq 1$, or with $w_0 = 1, w_n = (n!)^{-1},\ n \geq 1$, the product of the algebra is continuous with the uniform norm estimate*

$$\|a \circ b\| \leq \sqrt{3}\, \|a\|\, \|b\| . \tag{9}$$

As a more general class of norms we choose weights

$$w_0 = 1,\ w_n(\sigma, \rho, s) = (n!)^\sigma 2^{\rho n} (1+n)^s \text{ if } n \geq 1, \tag{10}$$

with real parameters σ, ρ, s. These weights satisfy the inequalities $w_n(\sigma_1, \rho_1, s_1) \leq w_n(\sigma_2, \rho_2, s_2)$ if $\sigma_1 \leq \sigma_2, \rho_1 \leq \rho_2, s_1 \leq s_2$. We denote by $\|a\|_{(\sigma,\rho,s)}$ the norm (4) defined with the weights $w_n(\sigma, \rho, s)$. The estimate (8) and the bounds $\frac{(m+n)!}{m!n!} \geq \frac{(2m)!}{(m!)^2} \geq const \cdot 2^{2m} m^{-\frac{1}{2}}$ if $n \geq m \geq 1$ and $1 \leq \frac{(1+m)(1+n)}{1+m+n} \leq 1 + \min(m, n)$ yield inequalities of the type (6) also for these norms. We obtain

$$(p+q-1) w_{p+q}(\sigma, \rho, s) \leq \delta w_p(\sigma', \rho', s') w_q(\sigma', \rho', s') \text{ if } p, q \geq 1 \tag{11}$$

with a constant $\delta \geq 1$ if $\sigma = \sigma' = -1$ with $\rho = \rho' \in \mathbb{R}$ and $s = s' \leq 0$, or if $\sigma = \sigma' < -1$ with $\rho = \rho' \in \mathbb{R}$ and $s = s' \in \mathbb{R}$.

The generalizations of (9) are therefore

$$\|a \circ b\|_{(-1,\rho,s)} \leq \sqrt{3}\, \|a\|_{(-1,\rho,s)} \cdot \|b\|_{(-1,\rho,s)} \text{ if } \rho \in \mathbb{R}, s \leq 0, \tag{12}$$

and

$$\|a \circ b\|_{(\sigma,\rho,s)} \leq \gamma\, \|a\|_{(\sigma,\rho,s)} \cdot \|b\|_{(\sigma,\rho,s)} \text{ if } \sigma < -1, \rho \in \mathbb{R}, s \in \mathbb{R}. \tag{13}$$

Here γ takes some value $\gamma \geq \sqrt{3}$ depending on the choice of the parameters σ and s.

Moreover, the inequalities (11) are valid for $(\sigma, \rho, s) \neq (\sigma', \rho', s')$ if $\sigma < \sigma'$ or if $\sigma = \sigma'$ and $\rho < \rho'$. The corresponding estimates for the norms are

$$\|a \circ b\|_{(\sigma,\rho,s)} \leq \gamma \|a\|_{(\sigma',\rho',s')} \cdot \|b\|_{(\sigma',\rho',s')} \text{ if } \sigma < \sigma' \text{ for all } \rho, \rho', s, s' \in \mathbb{R}, \tag{14}$$

and

$$\|a \circ b\|_{(\sigma,\rho,s)} \leq \gamma \|a\|_{(\sigma,\rho',s')} \cdot \|b\|_{(\sigma,\rho',s')} \text{ if } \rho < \rho' \text{ for all } \sigma, s, s' \in \mathbb{R}. \tag{15}$$

The value of $\gamma \geq \sqrt{3}$ depends on the choice of the parameters.

For the tensor algebra and for algebras of symmetrized tensors[5] the Hilbert space $\mathcal{A}_1 = \mathcal{H}$ generates the whole algebra. Given a (self-adjoint/positive) operator A on $\mathcal{H}$, the mapping $\Gamma(A)e_0 = e_0$ and $\Gamma(A)\,(f_1 \circ f_2 \circ ... \circ f_n) := (Af_1) \circ (Af_2) \circ ... \circ (Af_n)$ for $f_\mu \in \mathcal{H}, \mu = 1, ..., n$, and $n \in \mathbb{N}$, defines a unique (self-adjoint/positive) operator $\Gamma(A)$ on the algebra $\mathcal{A}$, which satisfies the relation

$$\Gamma(A)(a \circ b) = (\Gamma(A)a) \circ (\Gamma(A)b)\,. \tag{16}$$

The norms (4) with the weights (10) are then easily generalized to

$$\|a\|^2_{(\sigma,\rho,s)} = \sum_{n=0}^{\infty} (n!)^\sigma \left\|(\Gamma(A))^\rho a_n\right\|_n^2 (1+n)^s \text{ if } a = \sum_{n=0}^{\infty} a_n,\ a_n \in \mathcal{A}_n. \tag{17}$$

If A is an invertible positive operator with lower bound $A \geq 2 \cdot id$, then $\Gamma(A)$ satisfies $\left\|(\Gamma(A))^{-\rho}\, a\right\|_n \leq 2^{-n\rho} \|a\|_n$ for $a \in \mathcal{A}_n$ if $\rho \geq 0$. This bound and the relation (16) imply that the estimates (12),(13) and (15) are also valid for the norms (17), moreover (14) holds if $\rho \leq \rho'$.

If A^{-1} is a Hilbert-Schmidt operator then a family of norms (17) can be used to define a nuclear topology on the algebra $\mathcal{A}$. For the symmetric tensor algebra that has been done in the white noise calculus and in the Malliavin calculus, see e.g. [1] [11] [20]. For the algebra of antisymmetric tensors and for the superalgebras such nuclear topologies can be found in [12] and in [6]. But the estimates of these references are not strong enough to derive the results with a single Hilbert norm as presented in Corollary 2 and in eqs. (12) and (13).

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[5]This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.

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